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Splitting Algorithms for General Pseudomonotone Mixed Variational Inequalities

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Abstract. In this paper, we suggest and analyze a number of resolvent-splitting algorithms for solving general mixed variational inequalities by using the updating technique of the solution. The convergence of these new methods requires either monotonicity or pseudomonotonicity of the operator. Proof of convergence is very simple. Our new methods differ from the existing splitting methods for solving variational inequalities and complementarity problems. The new results are versatile and are easy to implement.

Key words: Variational inequalities; Resolvent equations; Iterative methods; Convergence; Fixed points

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1. Introduction

Variational inequality theory has emerged as an effective and powerful tool for studying a wide class of unrelated problems arising in various branches of regional, social, physical, engineering, pure and applied sciences in a unified and general framework. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications. An important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind, see [2–4, 6–8, 16, 19–29] and references therein. In recent years, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, Wiener-Hopf (normal) equations, linear approximation, auxiliary principle, and descent framework for solving variational inequalities and related optimization problems. It is well known that the projection methods and its variant forms and Wiener-Hopf equations technique cannot be used to suggest and analyze iterative methods for solving mixed variational inequalities due to the presence of the nonlinear term. These facts motivated us to use the technique of resolvent operators, the origin of which can be traced back to Martinet [14] and Brezis [3]. In this technique, the given operator is decomposed into the sum of two (or more) maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to develop very efficient methods, since one can treat each part of the original operator independently. The operator splitting methods and related techniques have been analyzed and studied by many authors including Peaceman and Rachford [30], Lions and Mercier [13], Glowinski and Le Tallec [9], and Tseng [37]. For an excellent account of the alternating direction implicit (splitting) methods, see Ames [1]. In the context of the mixed variational inequalities, Noor [16, 19–24] has used the resolvent operator technique to suggest some splitting type methods. A useful feature of the forward-backward splitting method for solving the mixed variational inequalities is that the resolvent step involves the subdifferential of the proper, convex and lower semicontinuous part only and the other part facilitates the problem decomposition.

Equally important is the area of mathematical sciences known as the resolvent equations, which was introduced by Noor [26]. Noor [26] has established the equivalence between the mixed variational inequalities and the resolvent equations using essentially the resolvent operator technique. The resolvent equations are being used to develop powerful and efficient numerical methods for solving the mixed variational inequalities and related optimization problems, see [15, 16, 19–26] and the references therein. It is worth mentioning that if the nonlinear term involving the mixed variational inequalities is the indicator function of a closed convex set in a Hilbert space, then the resolvent operator is equal to the projection operator. Consequently, the resolvent equations are equivalent to the Wiener-Hopf (normal) equations, which were introduced by Shi [34] and Robinson [31] in relation with the classical variational inequalities. It is now well known that the variational inequalities are equivalent to the Wiener-Hopf equations. This equivalence has played an important and significant part in developing various numerical methods for solving variational inequalities. For the recent applications of Wiener-Hopf equations, see [18, 25, 27, 31].

In this paper, we use the resolvent operator and resolvent equations technique to suggest and analyze a number of iterative methods. This paper is a continuation of our earlier works. First of all, we convey the basic ideas behind these iterative methods. It is well known that the problem (2.1) is equivalent to the fixed point problem of the form:

$$g(u) = J_{\varphi}[g(u) - \rho Tu], \tag{1.1}$$

where J_{φ} is the resolvent operator. Invoking this equivalence, Noor [16] has suggested and analyzed a number of resolvent methods for solving (2.1). For an invertible g, Equation (1.1) can be written as

$$g(u) = J_{\varphi}[g(u) - \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu]]. \tag{1.2}$$

Based on this formulation, Noor [24] proposed and analyzed another set of iterative methods for solving problem (2.1).

Using the updating technique of the solution, Equation (1.1) can be written as

$$g(u) = J_{\varphi}[J_{\varphi}[g(u) - \rho Tu] - \rho Tg^{-1}J_{\varphi}[[g(u) - \rho Tu] - \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu]]] . \tag{1.3}$$

This formulation was used to suggest and analyze another set of iterative methods for solving (2.1), see [23]. These results extend and generalize the splitting forward-backward methods of Peaceman and Rachford [30], Douglas and Rachford [5] and Tseng [37]. It is shown that the convergence of these splitting methods requires only monotonicity of the operator. Splitting methods in [23] are two steps forward-backward methods in which the order of T and $\partial\varphi$ has not been changed. It is worth pointing out that all these methods suggested in [16, 23, 24] differ from each other, since different formulations were used to suggest these methods.

In this paper, we again use the updating technique of the solution to suggest some three steps modified forward-backward splitting methods. These methods are comparable with the so-called θ-scheme of Glowinski and Le Tallec [9]. Here the order of T and $\partial\varphi$ has not been changed unlike in [9]. We consider the convergence criteria of these new methods. The convergence of three step forward-backward splitting methods require only the monotonicity of the operator, which is much weaker condition than the requirements for other splitting methods. Using the equivalence between the resolvent equations and the mixed variational inequalities, we suggest another method. The convergence of this method requires the pseudo-monotonicity, which is even weaker than the monotonicity of the operator. Consequently, our results represent an improvement and refinement of previously known results. It is interesting to compare the efficiency and practicality of the proposed methods with the other known methods and is the subject of future research.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi : H \rightarrow R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function.

For given nonlinear operators $T, g : H \rightarrow H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0 , \quad \text{for all } g(v) \in H . \tag{2.1}$$

The inequality of type (2.1) is called the general mixed variational inequality or the general variational inequality of the second kind [16, 23, 24]. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.1).

We remark that if $g \equiv I$, the identity operator, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u \rangle + \varphi(v) - \varphi(u) \geqslant 0 , \quad \text{for all } v \in H , \tag{2.2}$$

which are called the mixed variational inequalities. For the applications, numerical methods and formulations, see [2–4, 6–10, 17–21] and the references therein.

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u)\begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise}, \end{cases}$$

then the general mixed variational inequality (2.1) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geqslant 0\,, \quad \text{for all } g(v) \in K\,. \tag{2.3}$$

The inequality of the type (2.3) is known as the general variational inequality, which was introduced and studied by Noor [17] in 1988. It turned out that a class of unrelated odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied by the general variational inequality (2.3), see [18, 25–29].

If $K^* = \{u \in H : \langle u, v\rangle \geqslant 0,$ for all $v \in K\}$ is a polar (dual) cone of the convex cone K in H, the problem (2.3) is equivalent to finding $u \in H$ such that

$$g(u) \in K\,, \quad Tu \in K^*\,, \quad \langle g(u), Tu\rangle = 0\,, \tag{2.4}$$

which is known as the general complementarity problem. Note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem is known as the quasi(implicit) complementarity problem. If $g = I$, the identity operator, then problem (2.4) is the generalized complementarity problem, which has been studied extensively, see [2, 4, 6, 25, 29] and references therein.

For $g \equiv I$, the identity operator, the general variational inequality (2.3) collapses to: find $u \in K$ such that

$$\langle Tu, v - u\rangle \geqslant 0\,, \quad \text{for all } v \in K\,, \tag{2.5}$$

which is called the standard variational inequality, introduced and studied by Stampacchia [36] in 1964. For the recent state-of-the-art, see [2–4, 6–29, 30–37].

It is worth mentioning that the projection technique and its variant forms including the Wiener–Hopf equations cannot be used to suggest iterative methods for solving the (general) mixed variational inequalities of the types (2.1) and (2.2) due to the presence of the nonlinear term φ. To overcome this difficulty, one uses the resolvent operator technique to suggest some iterative methods for solving the problem (2.2). In this paper, we extend the resolvent operator technique for the general mixed variational inequality (2.1). For this purpose, we recall the following well known concepts and results.

DEFINITION 2.1 [3]. If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u)\,, \quad \text{for all } u \in H\,,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\| .$$

REMARK 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to \cup \{+\infty\}$ is a maximal monotone operator, we denote by

$$J_\varphi(u) = (I + \rho\, \partial\varphi)^{-1}(u) , \quad \text{for all } u \in H ,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H.

LEMMA 2.1 [3]. *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0 , \quad \text{for all } v \in H , \tag{2.6}$$

if and only if

$$u = J_\varphi z ,$$

where $J_\varphi = (I + \rho\, \partial\varphi)^{-1}$ is the resolvent operator and ρ is a constant. This property of the resolvent operator J_φ plays an important part in obtaining our results.

Let $R_\varphi \equiv I - J_\varphi$, where I is the identity operator and $J_\varphi \equiv (I + \rho\, \partial\varphi)^{-1}$ is the resolvent operator. For given nonlinear operators $T, g : H \to H$, consider the problem of finding $z \in H$ such that

$$Tg^{-1}J_\varphi z + \rho^{-1}R_\varphi z = 0 , \tag{2.7}$$

where $\rho > 0$ is a constant and g is invertible. The equations of the type (2.7) are called the general resolvent equations, see [16, 23, 24]. If $g \equiv I$, the identity operator, then the problem (2.7) reduces to: find $z \in H$ such that

$$TJ_\varphi z + \rho^{-1}R_\varphi z = 0 , \tag{2.8}$$

which are known as the resolvent equations, introduced and studied by Noor [26]. For the applications, formulation and numerical methods of the resolvent equations, see [19–25].

We remark that if φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently, problem (2.7) is equivalent to finding $z \in H$ such that

$$Tg^{-1}P_K z + \rho^{-1}Q_K z = 0 . \tag{2.9}$$

The equations of the type (2.9) are known as the general Wiener-Hopf equations, which are mainly due to Noor [18]. If $g \equiv I$, we obtain the original form of the Wiener-Hopf (normal map) equations, which were introduced and studied by Shi [34] and Robinson [31] independently. We would like to mention that the Wiener-Hopf equations technique is being used to develop some implementable and efficient iterative algorithms for solving variational inequalities and related fields. For the recent state-of-the-art, see [18, 25–27, 29, 33] and the references therein.

We also need the following concepts.

DEFINITION 2.2 For all $u, v \in H$, an operator $T : H \to H$ is said to be:

(i) *g-monotone*, if

$$\langle Tu - Tv, g(u) - g(v)\rangle \geqslant 0$$

(ii) *g-pseudomonotone*, if

$$\langle Tu, g(v) - g(u)\rangle \geqslant 0 \quad \text{implies} \quad \langle Tv, g(v) - g(u)\rangle \geqslant 0$$

(iii) *g-Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\langle Tu - Tv, g(u) - g(v)\rangle \leqslant \delta \|g(u) - g(v)\|^2 .$$

Note that for $g \equiv I$, the identity operator, Definition 2.2 reduces to the standard definition of monotonicity, pseudomonotonicity and (relaxed) Lipschitz continuity of the operator T. It is well known [6] that monotonicity implies pseudomonotonicity, but not conversely.

3. Main results

In this section, we suggest and analyze some new iterative methods for solving general mixed variational inequality (2.1). One can prove that the variational inequality (2.1) is equivalent to the fixed point problem by invoking Lemma 2.1.

LEMMA 3.1 [16]. *The function $u \in H$ is a solution of the mixed variational inequality (2.1) if and only if $u \in H$ satisfies the relation*

$$g(u) = J_\varphi[g(u) - \rho Tu] , \tag{3.1}$$

where $J_\varphi = (I + \rho\, \partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the general mixed variational inequality (2.1) is equivalent to the fixed point problem. This alternate equivalent formulation is very useful from the numerical point of view. This fixed point formulation enables us to suggest and analyze the following iterative algorithm.

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + J_\varphi[g(u_n) - \rho Tu_n] , \quad n = 0, 1, 2, \ldots .$$

For the convergence analysis of Algorithm 3.1, see Noor [16], if the operators T, g are both strongly monotone and Lipschitz continuous.

If g is invertible, then one can rewrite Equation (3.1) in the form

$$g(u) = J_\varphi[J_\varphi[g(u) - \rho Tu] - \rho Tg^{-1}J_\varphi[g(u) - \rho Tu]]$$
$$= J_\varphi[I - \rho Tg^{-1}]J_\varphi[I - \rho Tg^{-1}]g(u) .$$

This fixed point formulation allows us to suggest the following iterative method, which is known as the modified resolvent method.

ALGORITHM 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = J_\varphi[J_\varphi[g(u_n) - \rho Tu_n] - \rho Tg^{-1}J_\varphi[g(u_n) - \rho Tu_n]] ,$$
$$= J_\varphi[I - \rho Tg^{-1}]J_\varphi[I - \rho Tg^{-1}]g(u_n) , \quad n = 0, 1, 2, \ldots .$$

Algorithm 3.2 is a two step generalized forward-backward splitting method. Note that if gI, then Algorithm 3.2 is similar to the splitting method of Peaceman and Rachford [30]. For the convergence analysis of Algorithm 3.2, see Noor [23].

If g is invertible, then using the technique of updating the solution, Equation (3.1) can be written in the form

$$g(u) = J_\varphi[g(y) - \rho Ty] , \tag{3.2}$$

where

$$g(y) = J_\varphi[g(w) - \rho Tw] \tag{3.3}$$
$$g(w) = J_\varphi[g(u) - \rho Tu] . \tag{3.4}$$

From now onward, it is assumed that $g(y)$ and $g(w)$ are defined by the relations (3.3) and (3.4) respectively, unless otherwise specified.

We define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - J_\varphi[g(y) - \rho Ty] . \tag{3.5}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of the general mixed variational inequality (2.1) if and only if $u \in H$ is a zero of the equation

$$R(u) = 0 . \tag{3.6}$$

For a constant $\gamma \in (0, 2)$, Equation (3.6) can be written as

$$g(u) + \rho Tu = g(u) + \rho Tu - \gamma R(u) .$$

This formulation is used to suggest a new implicit method for solving the general mixed variational inequality (2.1).

ALGORITHM 3.3. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = g(u_n) + \rho Tu_n - \rho Tu_{n+1} - \gamma R(u_n), \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

We remark that if φ is the indicator function of a closed convex set K in H, then the resolvent operator $J_\varphi P_K$, the projection of H onto K. Consequently, the relation (3.5) becomes

$$R_K(u) = g(u) - P_K[g(y) - \rho Ty], \tag{3.8}$$

and Algorithm 3.3 collapses to Algorithm 3.4 for the general variational inequalities (2.3).

ALGORITHM 3.4. For a given $u_0 \in H$, $g(u_0) \in K$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = g(u_n) + \rho Tu_n - \rho Tu_{n+1} - \gamma R_K(u_n), \quad n = 0, 1, 2, \ldots.$$

If $g \equiv I$, the identity operator, then Algorithm 3.3 reduces to:

ALGORITHM 3.5 [19]. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n + \rho Tu_n - \rho Tu_{n+1} - \gamma R(u_n), \quad n = 0, 1, 2, \ldots.$$

where

$$R(u_n) = u_n - J_\varphi[y_n - \rho Ty_n]], \quad n = 0, 1, 2, \ldots.$$

If φ is the indicator function of a closed convex set K in H, then $J_\varphi P_K$, the projection of H onto K. Consequently Algorithm 3.5 collapses to:

ALGORITHM 3.6. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n + \rho Tu_n - \rho Tu_{n+1} - \gamma\{u_n - P_K[y_n - \rho Ty_n]\}, \quad n = 0, 1, 2, \ldots$$

which appears to be a new one for the variational inequalities (2.4).

If $\gamma = 1$, then Algorithm 3.3 collapses to:

ALGORITHM 3.7. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= (g + \rho T)^{-1}[J_\varphi[g(y_n) - \rho Ty_n] + \rho Tu_n] \\ &= (g + \rho T)^{-1}[J_\varphi[I - \rho Tg^{-1}]J_\varphi[I - \rho Tg^{-1}]J_\varphi[I - \rho Tg^{-1}] \\ &\quad + \rho Tg^{-1}]g(u_n), \\ n &= 0, 1, 2, \ldots. \end{aligned}$$

which is a splitting method and generalizes the modified forward-backward splitting methods of Tseng [37] and Noor [23].

For the convergence analysis of Algorithm 3.3, we need the following results.

LEMMA 3.2. *Let $\bar{u} \in H$ be a solution of (2.1). If $T: H \rightarrow H$ is a g-monotone operator, then*

$$\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u)\rangle \geqslant \|R(u)\|^2, \quad \textit{for all } u \in H. \tag{3.9}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1), then

$$\langle T\bar{u}, g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geqslant 0, \quad \text{for all } g(v) \in H. \tag{3.10}$$

Taking $g(v) = J_\varphi[g(y) - \rho Ty]$ in (3.10), we have

$$\rho\langle T\bar{u}, J_\varphi[g(y) - \rho Ty] - g(\bar{u})\rangle + \rho\varphi(_\varphi[g(y) - \rho Ty]) - \rho\varphi(g(\bar{u})) \geqslant 0. \tag{3.11}$$

Setting $z = g(u) - \rho Tu, u = J_\varphi[g(y) - \rho Ty], v = g(\bar{u})$ in (2.6), we obtain

$$\langle g(u) - \rho Tu - J_\varphi[g(y) - \rho Ty], J_\varphi[g(y) - \rho Ty] - g(\bar{u})\rangle$$
$$+ \rho\varphi(g(\bar{u})) - \rho\varphi(J_\varphi[g(y) - \rho Ty]) \geqslant 0. \tag{3.12}$$

Adding (3.11), (3.12) and using (3.5), we have

$$\langle R(u) - \rho(Tu - T\bar{u}), g(u) - g(\bar{u}) - R(u)\rangle \geqslant 0. \tag{3.13}$$

From (3.13), it follows that

$$\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u)\rangle \geqslant \langle R(u), R(u)\rangle + \rho\langle Tu - T\bar{u}, g(u) - g(\bar{u})\rangle$$
$$\geqslant \langle R(u), R(u)\rangle, \quad \text{since } T \text{ is } g\text{-monotone},$$

which implies that

$$\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u)\rangle \geqslant \|R(u)\|^2,$$

the required result. □

LEMMA 3.3. *Let $\bar{u} \in H$ be the solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.3, then*

$$\|g(u_{n+1}) - g(\bar{u}) + \rho(Tu_{n+1} - T\bar{u}\|^2$$
$$\leqslant |g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \tag{3.14}$$

Proof. Since $\bar{u}$ is a solution of (2.1) and u_{n+1} satisfies the relation (3.7), so

$$\|g(u_{n+1}) - g(\bar{u}) + \rho(Tu_{n+1} - T\bar{u})\|^2 = \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u}) - \gamma R(u_n)\|^2$$
$$\leqslant \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 - 2\gamma\|R(u_n)\|^2 + \gamma^2\|R(u_n)\|^2,$$

by using (3.9).

$$= \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2.$$ □

THEOREM 3.1. *Let $g: H \to H$ be invertible, then the approximate solution u_{n+1}*

obtained from Algorithm 3.3 converges to a solution $\bar{u}$ of the general variational inequality (2.1), provided that H is a finite dimensional space.

Proof. Let $\bar{u} \in H$ be a solution of (2.1). From (3.14), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leqslant \|g(u_0) - g(\bar{u}) + \rho(Tu_0 - T\bar{u})\|^2 ,$$

and consequently

$$\lim_{n\to\infty} R(u_n) = 0 .$$

Let $\bar{u}$ be the cluster point of $\{u_n\}$ and suppose the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u}$. Since $R(u)$ is continuous, so

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0 ,$$

and $\bar{u}$ is the solution of the general mixed variational inequality (2.1) by invoking Lemma 3.1 and

$$\|g(u_{n+1}) - g(\bar{u}) + \rho(Tu_{n+1} - T\bar{u})\|^2 \leqslant \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 .$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}) .$$

Since g is invertible, so

$$\lim_{n\to\infty} (u_n) = \bar{u} ,$$

which is the solution of the general mixed variational inequality. □

To implement Algorithm 3.3, one has to find the solution implicitly, which may create some problems. To overcome this difficulty, we suggest another iterative method, the convergence of which also requires only the monotonicity of the operator.

For a stepsize $\gamma \in (0, 2)$, Equation (3.6) can be written as

$$g(u) = g(u) - \gamma R(u) .$$

This fixed point formulation is used to suggest the following iterative method.

ALGORITHM 3.8. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} g(u_{n+1}) &= g(u_n) - \gamma R(u_n) \\ &= g(u_n) - \gamma\{g(u_n) - J_\varphi[g(y_n) - \rho T y_n]\} , \quad n = 0, 1, 2, \ldots . \end{aligned}$$

Note that for $\gamma = 1$, Algorithm 3.8 collapses to:

ALGORITHM 3.9. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = J_\varphi[g(y_n) - \rho T y_n]$$
$$= J_\varphi[I - \rho T g^{-1}] J_\varphi[I - \rho T g^{-1}] J_\varphi[I - \rho T g^{-1}] g(u_n), \quad n = 0, 1, 2, \ldots$$

which is three-step forward-backward splitting method and is a generalization of a so-called θ-scheme of Glowinski and Le Tallec [9], which they suggested by using the augmented Lagrangian technique. Note that the order of T and $\partial\varphi$ has not been changed. For related work, see [10] and the references therein.

If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K, and consequently Algorithms 3.8 and 3.9 reduce to the following algorithms respectively.

ALGORITHM 3.10. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = g(u_n) - \gamma\{g(u_n) - P_K[g(y_n) - \rho T y_n]\}, \quad n = 0, 1, 2, \ldots$$

ALGORITHM 3.11. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = P_K[g(y_n) - \rho T y_n]$$
$$= P_K[I - \rho T g^{-1}] P_K[I - \rho T g^{-1}] P_K[I - \rho T g^{-1}] g(u_n), \quad n = 0, 1, 2, \ldots$$

Following the technique of Theorem 3.1, one can easily show that the approximate solution u_{n+1} obtained from Algorithm 3.8 converges to the exact solution $\bar{u} \in H$ of the general variational inequality (2.1).

We now use the resolvent equation technique to propose another iterative method for solving the general mixed variational inequalities (2.1), the convergence of which requires the pseudomonotonicity of the operator. Using Lemma 2.1, Lemma 3.1 and the technique of Noor [16], we can establish the equivalence between the general mixed variational inequalities (2.1) and the resolvent equations (2.7). This equivalence is used to suggest a new iterative algorithm for solving the mixed variational inequality (2.1).

THEOREM 3.2. *The general mixed variational inequality (2.1) has a solution $u \in H$, if and only if the general resolvent equation (2.7) has a solution $z \in H$, where*

$$g(u) = J_\varphi z \tag{3.15}$$

and

$$z = g(y) - \rho T y, \tag{3.16}$$

where $\rho > 0$ is a constant.

Theorem 3.2 implies that the general mixed variational inequality (2.1) and the resolvent equations (2.7) are equivalent. We use this equivalence to suggest a new iterative algorithm for solving the general mixed variational inequalities (2.1).

Using the fact that $R_\varphi I - J_\varphi$, resolvent equations (2.7) can be written as

$$z - J_\varphi z + \rho T g^{-1} J_\varphi z = 0 .$$

Thus, for a stepsize γ, we can write as

$$\begin{aligned} g(u) &= g(u) - \gamma\{z - J_\varphi z + \rho T g^{-1} J_\varphi z\} \\ &= g(u) - \gamma d , \end{aligned}$$

where

$$d = R(u) - \rho T u + \rho T g^{-1} J_\varphi [g(y) - \rho T y] .$$

This fixed point formulation allows us to suggest the following iterative algorithm for solving general mixed variational inequalities (2.1).

ALGORITHM 3.12. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} w_n &= J_\varphi[g(u_n) - \rho T u_n] \\ y_n &= J_\varphi[g(w_n) - \rho T w_n] \\ z_n &= g(y_n) - \rho T y_n \\ d_n &= z_n - J_\varphi z_n + \rho T g^{-1} J_\varphi z_n \\ g(u_{n+1}) &= g(u_n) - \gamma d_n , \quad n = 0, 1, 2, \ldots \end{aligned}$$

We note that if $g \equiv I$, the identity operator, then Algorithm 3.12 collapses to the following new algorithm for solving the mixed variational inequalities (2.2).

ALGORITHM 3.13. For a given $u_0 \in H$, compute the approximate solution

$$\begin{aligned} w_n &= J_\varphi[u_n - \rho T u_n] \\ y_n &= J_\varphi[w_n - \rho T w_n] \\ z_n &= y_n - \rho T y_n \\ d_n &= z_n - J_\varphi z_n + \rho T J_\varphi z_n \\ u_{n+1} &= u_n - \gamma d_n , \quad n = 0, 1, 2, \ldots \end{aligned}$$

In brief, for a suitable and appropriate choice of the operators, T, φ and the space H, one can obtain a number of algorithms for solving various classes of variational inequalities and the related optimization problems. For the convergence analysis of Algorithm 3.12, we need the following results.

LEMMA 3.4. *Let $\bar{u} \in H$ be a solution of (2.1) and $T : H \to H$ be a g-pseudo-monotone and g-Lipschitz continuous operator with a constant $\delta > 0$. Then*

$$\langle g(u) - g(\bar{u}), R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty]\rangle$$
$$\geqslant \{1 - \rho\delta\}\|R(u)\|^2, \quad \textit{for all } u \in H. \tag{3.17}$$

Proof. Since T is g-pseudomonotone, for all $v, \bar{u} \in H$, so, from (3.10), we have

$$\langle Tv, g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0. \tag{3.18}$$

Taking $g(v) = J_{\varphi}[g(y) - \rho Ty]$ in (3.18), we have

$$\langle Tg^{-1}J_{\varphi}[g(y) - \rho Ty], J_{\varphi}[g(y) - \rho Ty] - g(\bar{u})\rangle$$
$$+ \varphi(J_{\varphi}[g(y) - \rho Ty]) - \varphi(g(\bar{u})) \geqslant 0. \tag{3.19}$$

Adding (3.12) and (3.19), we have

$$\langle g(u) - g(\bar{u}), R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty]\rangle$$
$$\geqslant \langle R(u), R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty]\rangle. \tag{3.20}$$

Since T is a g-Lipschitz continuous operator with a constant $\delta > 0$, so

$$\langle Tu - Tv, g(u) - g(v)\rangle \leqslant \delta\|g(u) - g(v)\|^2. \tag{3.21}$$

From (3.5), and (3.21), we obtain

$$\langle R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty], R(u)\rangle\|R(u)\|^2$$
$$- \rho\langle Tu - \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty], R(u)\rangle \geqslant \{1 - \rho\delta\rangle\|R(u)\|^2. \tag{3.22}$$

Combining (3.20) and (3.22), we have

$$\langle g(u) - g(\bar{u}), R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(y) - \rho Ty]\rangle \geqslant (1 - \rho\delta)\|R(u)\|^2,$$

the required result.

LEMMA 3.5. *The sequence $\{u_n\}$ generated by Algorithm 3.12 for general mixed variational inequalities (2.1) satisfies the inequality*

$$\|(g(u_{n+1}) - g(\bar{u})\|^2 \leqslant \|g(u_n) - g(\bar{u})\|^2\gamma(1 - \delta\rho)(2 - \gamma(1 - \delta\rho))\|R(u_n)\|^2,$$
for all $\bar{u} \in H$.

Proof. From (3.15), and Algorithm 3.12, we have

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \; \|g(u_n) - g(\bar{u}) - \gamma d_n\|^2$$
$$\leqslant \|g(u_n) - g(\bar{u})\|^2 - 2\gamma\langle g(u_n) - g(\bar{u}), d_n\rangle + \gamma^2\|d_n\|^2$$
$$\leqslant \|g(u_n) - g(\bar{u})\|^2 - 2\gamma\langle R(u_n), d_n\rangle + \rho^2\|d_n\|^2$$
$$\leqslant \|g(u_n) - g(\bar{u})\|^2 - \gamma(1 - \rho\delta)(2 - \gamma(1 - \rho\delta))\|R(u_n)\|^2.$$ □

THEOREM 3.3. *Let $\{u_n\}$ be the approximate solution obtained from Algorithm 3.12 and $\bar{u} \in H$ be a solution of (2.1), then* $\lim_{n\to\infty}(u_n) = \bar{u}$.

Proof. Its proof follows from Theorem 3.1.

4. Conclusion

We have suggested and analyzed a number of new iterative methods for solving general mixed variational inequalities by using the technique of updating the solution. Convergence of some of these methods requires the pseudomonotonicity of the operator, which is weaker than the monotonicity. In this respect, our results represent an improvement and refinement of the previous results. The comparison of these new methods with the other standard techniques for solving the general variational inequalities is an interesting problem for further research.

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